Factorization Methods

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1 Methods

Let's recall briefly the notion of factorization by applying it to numbers. When one's asked to *break a number down to prime factor,* it means that the task is to rewrite the number as *product of prime numbers* (possibly raised to powers). For instance, consider the number 432. To get the factorization we have to *divide* the number by the first prime number we can think of, that is 2. In this way we have

$$432 = 2 \cdot 216$$

The resulting number, 216, is still divisible by 2, so we have

$$432 = 2 \cdot 2 \cdot 108 = 2^2 \cdot 108$$

Proceeding in this way we have in the end

$$432 = 2^2 \cdot 2 \cdot 56 = 2^3 \cdot \underbrace{2 \cdot 28}_{=56} = 2^4 \cdot \underbrace{2 \cdot 14}_{=28} = 2^5 \cdot \underbrace{2 \cdot 7}_{=14} = 2^6 \cdot 7$$

Since 7 is a prime number, the factorization is over and we have written 432 as the product of two prime numbers: 2 (with power 6) and 7.

With polynomials (that are algebraic expressions containing both letters and numbers, joined by the usual operations of addition, multiplication end power raising) the principle is basically the same: *factoring a polynomials means rewrite it as the product of polynomials of less degree, in such a way that the new polynomials are no longer factored.* Let's explain this last statement. The polynomial

$$x^4 - 1$$

can be factored in the product

$$x^4 - 1 = (x^2 + 1)(x^2 - 1)$$

but the resulting decomposition is *not* a prime factor decomposition since the second polynomial can be further factored:

$$x^4 - 1 = (x^2 + 1)(x + 1)(x - 1)$$

In the following pages we sum up the most common methods to factor a polynomial.

1. **Total recollection**
   This method is based on the search of the *greates common divisor* (G.C.D.) among the terms of the polynomials. Once you have found it, you can *collect* the term by dividing the polynomial by the G.C.D.

   $$12x^3y - 4x^2y^2 + 16xy^3$$

   By inspection the polynomial we can see that the number 4 and the letters x and y show up in eact term of the polynomial. Since finding the G.C.D. means to consider only common letters with highest degree, we can see that the common factor among the three terms is $4xy$. Collecting it we obtain

   $$4xy(3x^2 - xy + 4y^2)$$

The polynomial in brackets is the *result of the division of* $12x^3y - 4x^2y^2 + 16xy^3$ *by* $4xy$.

2. **Partial recollection**
   In some cases it's not possibile to perform a total recollection among *all* the terms of the polynomial. But, if the polynomial has and even number of terms (and greater or equato to 4) then it is possible to collect a common factor between two terms and then a different common term between the remaining two terms. After that, it should be possibile to perform a total recollection. For instance, consider the polynomial

$$ax + ay + bx + by$$

   We can collect a between the first and the second term, and b between the third and the fourth. In such a way we obtain

$$a(x + y) + b(x + y)$$

   Now we are again in the case of total recollection: we have transformed the 4-terms polynomial in a 2-terms polynomial with the common factor $(x + y)$. Collecting it we end up with

$$(x + y)(a + b)$$

   Note that the same result could be obtained collecting x between first and third term, and y between second and fourth:

$$x(a + b) + y(a + b) = (x + y)(a + b)$$

3. **Remarkable products**
   When both total and partial recollation fail, we can try to see if the polynomial can be brought back to one of the following identities, known as *remarkable products*.

$$\begin{aligned}
A^2 - B^2 &= (A + B)(A - B) \\
A^2 + 2AB + B^2 &= (A + B)^2 \\
A^2 - 2AB + B^2 &= (A - B)^2 \\
A^3 + 3A^2B + 3AB^2 + B^3 &= (A + B)^3 \\
A^3 - 3AB^2 + 3AB^2 - B^3 &= (A - B)^3 \\
A^3 + B^3 &= (A + B)(A^2 - AB + B^2) \\
A^3 - B^3 &= (A - B)(A^2 + AB + B^2)
\end{aligned}$$

   Observe that A and B could be *complex expression of numbers and letters*, and not necessarirly single letters. For instance, in the polynomial

$$(x + y)^2 + 2(x + y)(3x - 5y) + (3x - 5y)^2$$

   setting

$$\begin{aligned}
A &= x + y \\
B &= 3x - 5y
\end{aligned}$$

   we are brought back to identity

$$A^2 + 2AB + B^2 = (A + B)^2$$

   so that we can write

$$\underbrace{(x + y)^2}_{A^2} + \underbrace{2(x + y)(3x - 5y)}_{2AB} + \underbrace{(3x - 5y)^2}_{B^2} = (\underbrace{x + y}_{A} + \underbrace{3x - 5y}_{B})^2 = (4x - 4y)^2 = 16(x - y)^2$$

4. **Second order 3-terms polynomial**
   This method can be applied to factor polynomials of the form

$$ax^2 + bx + c$$

   where a, b, c are real numbers. We can face it in two ways:

   (a) If $a = 1$, that is the polynomial is of the form $x^2 + bx + c$, we can try to find two numbers x_1 and x_2 such that

$$\begin{aligned} x_1 + x_2 &= -b \\ x_1 x_2 &= c \end{aligned}$$

   In this way the decomposition can be written as

$$x^2 + bx + c = (x - x_1)(x - x_2)$$

   (the method can be applied also if $a \neq 1$, but it is a bit more difficult, so we apply directly the way at point b).

   (b) In any case (both $a = 1$ or $a \neq 1$), we can find x_1 and x_2 by applying the second order equation formula

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   and write

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

   (remember to put the number a in front of everything!).

5. **Ruffini's Rule**
   The last method (that is the one to be used if none of the provious ones works!) is known as *Ruffini's Rule* and it's based on the fact (*Ruffini's Theorem)* that if $P(x)$ is a polynomial and a a real number such that $P(a) = 0$, then $P(x)$ is can be divided by the polynomial $x - a$. Therefore the algorithm used to factor a polynomial with Ruffini's rule can be summarized in this way:

   (a) Identify the number a such that $P(a) = 0$. Usually this number can be found among fractions of the form $\frac{m}{n}$ where n is a divider of the knwon term of the polynomial, and m is a divider of the coefficient of the highest degree term.

   (b) Perform the division of $P(x)$ by $x - a$ and obtain the quotient $Q(x)$. In this way the decomposition can be written as

$$P(x) = (x - a)Q(x)$$

   (c) Perform again points (a) and (b) if necessary.

2 Examples

1. $3x^3 + 3xy^2 + 2x^2y + 2y^3 + x^2z + y^2z$
   Since the polynomial has 6 terms and there is no common factors in the whole expression, we can try to perform a partial collection. For instance, collecting $3x$ among the first two terms, $2y$ among the third and the fourth, and z among the last two, we get

$$3x(x^2 + y^2) + 2y(x^2 + y^2) + z(x^2 + y^2)$$

   Now we have put in evidence the common factor $x^2 + y^2$: collecting it we have in the end

$$(x^2 + y^2)(3x + 2y + z)$$

2. $9z^8 - 25t^2$

   Of course we cannot apply any kind of collection. Inspecting the remarkable products at point (3) we can factor the polynomial using the *squares' difference* formula:

$$9z^8 - 25t^2 = (3z^4 - 5t)(3z^4 + 5t)$$

3. $25x^2 + 20xy + 4y^2$

   Also in this case we can find the proper formula among the remarkable products:

$$\underbrace{25x^2}_{A^2} + \underbrace{20xy}_{2AB} + \underbrace{4y^2}_{B^2} = (\underbrace{5x}_{A} + \underbrace{2y}_{B})^2$$

4. $5x^2 - 18x - 8$

   In this case the polynomial cannot fit any of the formulae of remarkable products, thus we try to apply the second-order equation formula to get

$$x_{1,2} = \frac{18 \pm \sqrt{18^2 + 4 \cdot 8 \cdot 5}}{2 \cdot 5} = \frac{18 \pm \sqrt{324 + 160}}{10} = \frac{18 \pm 22}{10}$$

   Therefore

$$x_1 = 4 \qquad x_2 = -\frac{2}{5}$$

   and the decomposition is

$$5\,(x-4)\left(x + \frac{2}{5}\right) = (x-4)(5x+2)$$

5. $x^4 + 2x^3 - 7x^2 - 2x + 6$

   Now is Ruffini's rule time. First of all we have to find the number a such that $P(a) = 0$ (that is, by substituting x with a in the polynomial we get zero). This number, as stated before, can be found among

$$\pm 1, \pm 2, \pm 3, \pm 6$$

   We try first with 1 and -1:

$$\begin{aligned} P(1) &= 1 + 2 - 7 - 2 + 6 = 0 \\ P(-1) &= 1 - 2 - 7 + 2 + 6 = 0 \end{aligned}$$

   Since both $P(1)$ and $P(-1)$ are zero, the polynomial si divisible by $(x-1)$ and $(x+1)$. Performing the division we finally get

$$(x+1)(x-1)(x^2 + 2x - 6)$$